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Statistics of undeveloped speckles in partially polarized light

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Abstract. We obtain an expression for the probability density function (PDF) of partially developed speckles formed by light with an arbitrary degree of polarization. From the probability density we calculate the detection threshold corresponding to the 5σ confidence level of a normal distribution. We show that unpolarized light has an advantage in high contrast imaging for low ratios of the deterministic part of the point spread function (DL PSF) to the halo, typical in coronagraphy.

1 Introduction

The speckle phenomena in laser optics created a great interest about 40 years ago. Now this interest grows gain in the new context of high contrast astronomical imaging. Imaging and characterization of extra-solar planets close to their parent star is extremely difficult, because it involves detection at very low signal to noise ratios. One of the main contributors to the noise is quasi-static speckles of the stellar light, coursed by the imperfections in the imaging optics. They do not merge into a smooth background with long exposures, but stay as localized spots of light very similar to any possible planet. The knowledge of the statistical properties of these speckles is essential for developing smart detection algorithms. We do not address the detection itself, but the statistics of stellar speckles.

A number of authors have used a modified Rician distribution for the PDF of stellar speckles [1],[2],[3]. This distribution was obtained assuming fully polarized light [4], [5], so the results cannot be applied directly to the case of astronomical images. The light from a star is known to be unpolarized [6]. Imaging optics introduces some polarization [7], so in general the speckles from the star arriving to the detector are partially polarized.

We calculated the probability density function (PDF) for the undeveloped speckle field in a case of arbitrary degree of polarization, P [8]. *Undeveloped speckle* field term is used to describe speckles appeared on the constant background - diffraction limit core of the star image. The intensity of the diffraction limit core related to the mean level of the speckled component is the second main parameter of our equations, r. Parameter r can take a waist range of values. Typically, for a systems without a coronagraph and with adaptive optics of moderate Strehl ratios: $r \sim 1 - 10$. The diffraction limit core can be to some extend removed by a coronagraph, which means that r can take quite small values.

From the PDF we calculate the detection threshold corresponding to a given confidence level. For the later we take a level corresponding to the 5σ confidence level of a normal distribution: $CL = 0.9999997125$. We show that the threshold depends on the degree of polarization, as well as on the factor r.

2 Background

We consider a wave emitted by a star, passing through a turbulent atmosphere, a telescope, a correcting system, entering a pupil of an imaging system. Although the light from the star

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is unpolarized, it can be subjected to some polarization while propagating to the imaging pupil. We consider a partially polarized light with an arbitrary degree of polarization, P. Unpolarized and fully polarized light are two particular cases with $P = 0$ and $P = 1$ correspondingly. Suppose that we observe through a narrow filter ($\Delta\lambda/\lambda << 1$), so the wave is quasi-monochromatic. The total vector field is composed of two linearly polarized components [9]. The polarization properties are described by a coherency matrix. A wave with correlated $x-$ and $y-$polarization components is equivalent to a wave with uncorrelated polarization components having different values of intensity in the two components, but the same sum of the two intensities:

$$I = I_1 + I_2. \tag{1}$$

Based on this fact, the probability density function for fully developed speckles in partially polarized light was obtained [4], [5], using a fundamental result from probability theory. It states that the probability density function of the sum of independent random variables is a convolution of the probability density functions of the components. We are exploring this approach to obtain a probability density for the undeveloped speckles. Our goal is to obtain a most general expression from which all known particular cases can be deduced.

3 Probability density function for partially polarized light

3.1 General formula

Each of the intensity in Eq.(1) patterns is a sum of squares of real and imaginary parts of the complex amplitude, for which we use a classical assumption of a non-central Gaussian distribution. I_1 and I_2 follow a non-central chi-square distribution of the second order. For example, for the first component the PDF is

$$p(I_1) = \frac{1}{H_1} \exp\big(-\frac{I_1 + D_1}{H_1}\big) I_0\big(2\frac{\sqrt{I_1 D_1}}{H_1}\big), \tag{2}$$

where I_0 is a modified Bessel function of zero order. The parameters of the distribution, H and D, depend on the degree of polarization as:

$$D_{1,2} = \frac{1}{2}(1 \pm P)|C(\mathbf{w})|^2, \tag{3}$$
$$H_{1,2} = \frac{1}{2}(1 \pm P)|S(\mathbf{w})|^2.$$

The *plus* is taken for the speckle pattern I_1 and *minus* is taken for the speckle pattern I_2. Eq.2 is also known as a modified Rician distribution. The sum $D_1 + D_2 = |C(\mathbf{w})|^2$ is the DL PSF, the deterministic part of the PSF defined by the optical system without random disturbances. It can be also a PSF reduced to some extend by a coronagraph. The sum $H_1 + H_2 = |S(\mathbf{w})|^2$ is the halo, an ensemble average speckled field on top of the DL PSF. Both functions depend on the coordinate on the image plane $\mathbf{w}$. The mean level of the total intensity equals to the DL PSF plus the halo: $\langle I\rangle = |C(\mathbf{w})|^2 + |S(\mathbf{w})|^2$. To simplify the look of the forthcoming equations we move to the normalized intensities:

$$x_{1,2} = \frac{I_{1,2}}{\langle I\rangle}, \tag{4}$$

and introduce parameter r - the ratio between the DL PSF and the halo:

$$r = \frac{|C(\mathbf{w})|^2}{|S(\mathbf{w})|^2}. \tag{5}$$

For the ideal coronagraph entirely removing the DL PSF $r = 0$.

We write the PDF for the normalized intensities:

$$p_{1,2}(x) = 2\frac{1+r}{1\pm P}\exp\big[-\frac{2x(1+r)}{1\pm P} - r\big] I_0\big[2\sqrt{\frac{2x(1+r)r}{1\pm P}}\big], \tag{6}$$

where *plus* is taken for p_1 and *minus* for p_2. The PDF of the total field can be calculated either by numerical convolution of p_1 and p_2, or by the method of the characteristic function [11]. The characteristic functions for the PDFs from Rician distributions are

$$\Xi_{1,2}(z) = \exp\big[\frac{r}{1 - iz\frac{1\pm P}{2(1+r)}} - r\big]\frac{1}{1 - iz\frac{1\pm P}{2(1+r)}}, \tag{7}$$

where, again, *plus* is taken for Ξ_1 and *minus* for Ξ_2. The PDF of the total normalized intensity, $x = I/\langle I\rangle$, is the inverse Fourier transform of the product $\Xi_1\Xi_2$. To proceed with the calculations we expand each of the exponential functions in Eq.7 in a sum and use the following table integral:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(\alpha - iz)^{-(l+1)}(\beta - iz)^{-(k+1)}\exp(-izx)dx \tag{8}$$
$$= \exp(-\alpha x)\frac{x^{k+l+1}}{\Gamma(k+l+2)}M[k+1, k+l+2, (\alpha-\beta)x],$$

where $M[...]$ is the degenerated hypergeometric function, also known as Kummer function, $\Gamma(...)$ is the Gamma function [13]. After re-combination of terms the final expression for the PDF becomes:

$$p(x) = 4x\frac{(1+r)^2}{1-P^2}\exp\big\{-\frac{2}{1-P}\big[x(1+r) + r(1-P)\big]\big\} \tag{9}$$
$$\sum_{l,k=0}^{\infty}\frac{1}{\Gamma(k+l+2)k!l!}\big[\frac{2r(1+r)x}{1-P}\big]^l\big[\frac{2r(1+r)x}{1+P}\big]^k$$
$$M\big[k+1, k+l+2, \frac{4xP(1+r)}{1-P^2}\big].$$

This is the general expression for the PDF for the arbitrary values for r and P. It is plotted in Figure 1 on example when the DL PSF is equal to the halo ($r = 1$). When the degree of polarization decreases the distribution becomes more localized around the mean, which indicates that the fluctuation of speckles decreases. It comes from the representation of partially polarized light as the sum of two independent speckle patterns. If two or more speckle patterns are added up, the total field is smoothed out, and the intensity fluctuation is lower. The difference is essential for the points in the image plane where $r \leq 1$. This situation is met in coronagraphic imaging. For observations without a coronagraph $r \gg 1$ in all areas interesting for the detection and the gain is not significant. In this regime, the distribution looks similar to a Gaussian.

3.2 Particular cases

For the developed speckle field the specular component is absent and $r = 0$. In this case in the double sum of Eq.9 all terms equal zero except for the first one: $l = 0$ and $k = 0$. Kummer function $M[1, 2, z]$ can be expressed through the exponential function [13]: $zM[1, 2, z] = e^z - 1$. After this substitution Eq.9 transforms into the PDF in the well-known form:

$$p(x, r=0) = \frac{1}{P}\big[\exp\big(-\frac{2x}{1+P}\big) - \exp\big(-\frac{2x}{1-P}\big)\big] \tag{10}$$

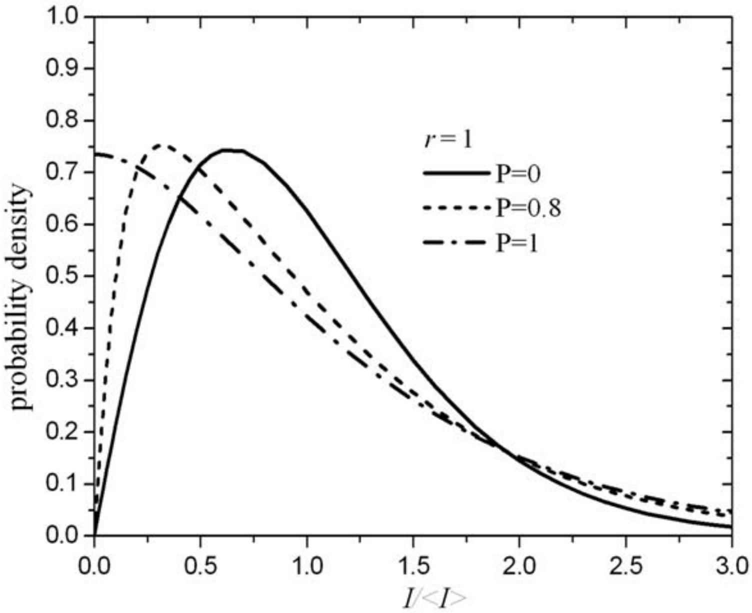

Fig. 1. PDFs for different values of the degree of polarization when the DL PSF is equal to halo

The second particular case of the fully polarized light answers condition $P = 1$. We use the following limit:

$$M[a, b, z] =_{z\to\infty} \frac{\Gamma(a)}{\Gamma(b)} e^z z^{a-b} \tag{11}$$

for any a and b. This allows factorizing the double sum in Eq.9, which converges to Rician distribution:

$$p(x, P = 1) = (1 + r) \exp[-(1 + r)x - r] I_0[2\sqrt{r(1 + r)x}] \tag{12}$$

Third particular case is when $P = 0$, i.e. the case of unpolarized light. Eq.1 is a sum of squares of four identically distributed normal values, hence the distribution in this case must coincide with the chi-square distribution of the fourth order. The proof is based on the fact that for any a and b: $M[a, b, 0] = 1$. Afterwards we apply a multiplication theorem for the Bessel functions ([13]). The PDF in this case is

$$p(x, P = 0) = 4(1 + r)^2 2x \exp[-2(1 + r)x - 2r] \frac{I_1[4\sqrt{r(1 + r)x}]}{4\sqrt{r(1 + r)x}}, \tag{13}$$

where I_1 is a modified Bessel function of degree one.

4 Detection threshold

Provided the knowledge about the PDF we now calculate the detection threshold corresponding to a given confidence level. For Gaussian statistics, the conventional 5σ level answers to a $CL_{5\sigma} = 0.9999997125$ confidence level. It means that if the statistics of speckle intensity were Gaussian, a probability of finding a speckle with an intensity higher than $I_{th} = \langle I\rangle + 5\sigma_I$ would be equal to $1 - CL_{5\sigma} = 2.875 \cdot 10^{-7}$. The application of the 5σ rule to the probability density found in Eq.9 leads to a lower confidence level. We now wish to find a detection threshold $n(P, r)$ for this PDF providing the same confidence level. For the normalized intensity it

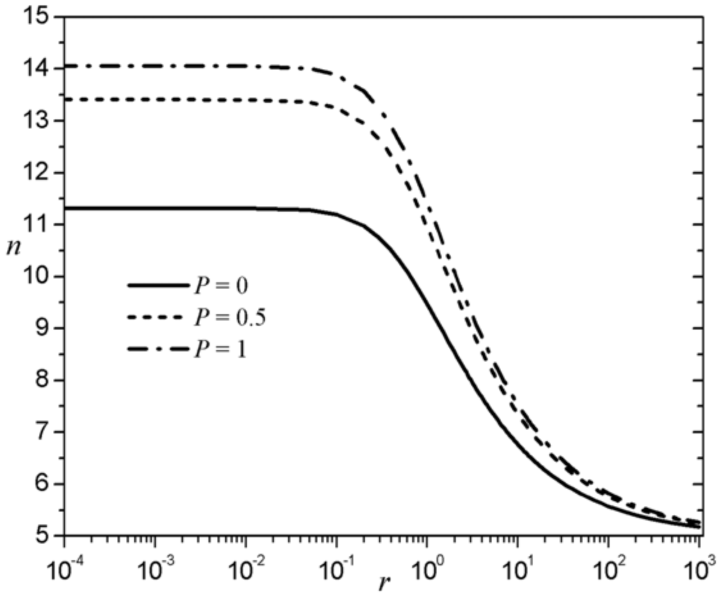

Fig. 2. Detection threshold as a function of the ratio between the DL PSF and the halo. Curves are shown for three different values of the degree of polarization

means:

$$prob[x > n(P,r)\sigma_x(P,r)] = 1 - CL_{5\sigma}. \quad (14)$$

The standard deviation can be calculated exactly as a square root of a sum of two variances for uncorrelated x_1 and x_2:

$$\sigma_x(P,r) = \sqrt{\frac{1+P^2}{2}}\frac{\sqrt{1+2r}}{1+r}. \quad (15)$$

We calculate the function $n(P,r)$ numerically from Eq.9. It is shown in Figure 2. When $r \ll 1$, a threshold $n = 14$ must be considered for fully polarized light and $n = 11$ for unpolarized light. When $r \sim 1$ there is still a difference in threshold: $n = 11$ for $P = 1$ and $n = 9$ for $P = 0$. Therefore, for small and medium values of r a lower threshold is required to reach the same confidence level of detection with unpolarized light, compared to polarized light. For large r the threshold saturates to the level $n = 5$ corresponding to the normal distribution.

For the high contrast imaging (HCI) the value of r reflects the performance of a coronagraph meant to reduce the DL PSF with respect to the performance of extreme adaptive optics (XAO) meant to decrease the halo. Small r (at a given angular distance) means that the coronagraph outperforms the XAO; large r means the opposite. To help designing a HCI instrument for any predicted performance of coronagraph and XAO we find an analytical expression approximating the curves from Figure 2. It estimates the detection threshold for arbitrary r and arbitrary degree of polarization required to achieve a confidence level equivalent to the 5σ level of Gaussian distribution:

$$n_x(P,r) = [n(0,P) - 5]\frac{\sqrt{1+2.4(2r)}}{1+2.4r} + 5. \quad (16)$$

Here $n(0,P)$ is the threshold in the case of fully developed speckles (Figure 3).

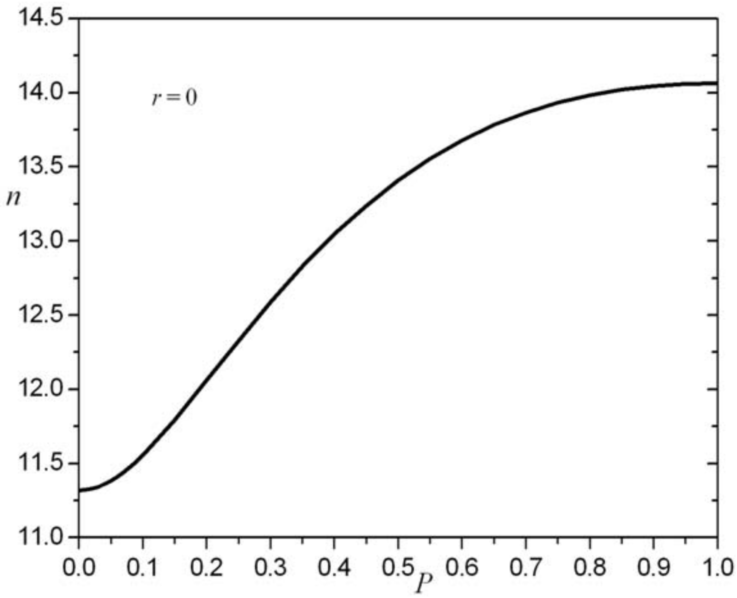

Fig. 3. Detection threshold as a function of the degree of polarization in absence of the DL PSF

Our study has been within the model of non-central Gaussian speckles. There are many effects causing speckle statistics to deviate from a non-central Gaussian: a finite spectral bandwidth, a finite pixel size on the detection device, or a limited number of field contributions. These facts will smooth the difference between the cases of polarized and unpolarized light.

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